Orthogonally Stiffened Cylindrical Shells Subjected to Internal Pressure

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Comprehensive investigation of two configurations of stiffened cylindrical shells subjected to internal pressure are made and illustrated by C-5A fuselage-type examples. One configuration has equally spaced longitudinal stringers and circumferential rings, while the other has uniform circumferential reinforcing bands (straps) added between every two rings. The skin, stringers, and rings are treated as distinct individual elements. The interactions among the elements are coupled by requiring compatible deformations along all element intersections. Numerical results for stiffened shells with, and without, reinforcing circumferential straps based on C-5A fuselage material parameters and dimensions are shown as plots of displacements and of inner and outer fiber stresses at various locations. The plotted results show that the straps, which are located midway between every two rings, used on C-5A fuselage are highly effective in equalizing stress levels throughout the stiffened shell, with associated reduction of maximum stress levels. This reduction will produce significant benefits in fatigue strength performance where fuselage pressurization and depressurization are predominant, and in resistance to damage propagation.

Nomenclature

a = shell radius
A_R = ring cross-sectional area
b = stringer spacing
D_i = plate rigidities of skin
e_{ij} = strain tensor
E = modulus of elasticity
F = Airy stress function
I = moment of inertia of the stringer cross-sectional area
I_R = moment of inertia of the ring cross-sectional area
K_{ij} = curvature tensor
l = ring spacing
M_s = bending moment in stringer
M_{ij} = stress couples in skin
M_r = bending moment in ring
N_{ij} = stress resultants in skin
p = internal pressure
P = skin normal loading function (along outward normal is positive)
$\bar{P}$ = $\pi a^2 p$ = axial load
q = load between skin and stringer
$2Q$ = load between skin and ring
R_0 = ring mid-surface radius
T_r = normal stress resultant in ring
t_i = effective shell thicknesses in skin material
V_s = transverse shear in stringer
V_r = transverse shear in ring
v = circumferential ring displacement
w = radial skin displacement (along inward normal is positive)
W_s = stringer displacement
W_r = radial ring displacement (along inward normal is positive)
x,y = longitudinal and circumferential coordinates
ν = Poisson's ratio, dimensionless
X_{ij} = change of curvature tensor
λ,μ = Lamé constants

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Introduction

CYLINDRICAL shells stiffened by longitudinal stringers and rings have been used widely for various structural purposes. Due to the complexity of the configuration the structure has been analyzed in the past by considering an equivalent homogeneous orthotropic shell with effective extensional and flexural stiffnesses as may be seen in almost all references.[1–9] The discussion on the determination of the rigidity properties may be referred to in Refs. 10, 11. Such idealization yields results which are quite acceptable for general stability and free vibration analyses when only the buckling load and the natural frequencies are needed, respectively. If one wishes to investigate the actual behavior of the structure or to design the structure in order to provide adequate strength or fatigue endurance, it becomes necessary to know the actual deformation of the shell and the actual stress distribution in the structure, and the analysis based on the idealization of the structure to that of an equivalent homogeneous orthotropic shell becomes undesirable. A number of authors[12–15] have investigated analytically cylindrical shells stiffened by elastic rings and subjected to uniform pressure. Since the loading and geometry are symmetrical about the longitudinal axis, the problems become one-dimensional. Howland and Beed[16] tested cylindrical shells stiffened orthogonally by equally spaced stringers and rings subjected to internal pressure. However, no flexural rigidities of stiffeners and their spacings are given. Bartolozzi[17] has discussed the general solution for the free vibrations of longitudinally stiffened cylindrical shells by treating the shell and stringers as individual components. No numerical example is given. Egle and Sewall[18] have studied the free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements. Beam modal functions are used to represent the deformation of the stringer as well as the shell in longitudinal direction, and Rayleigh-Ritz procedure is utilized. The present study is

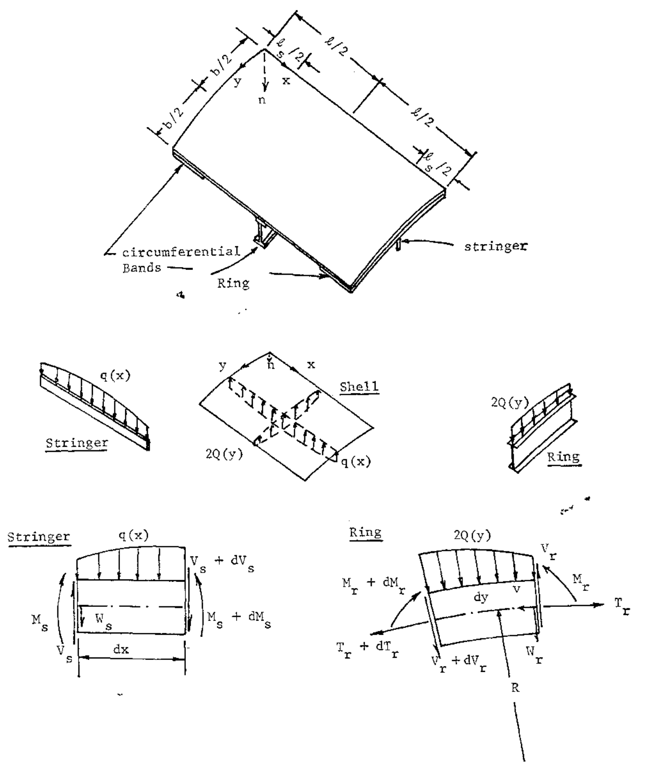

Fig. 1 Typical sections of stiffened shell structure.

concerned with the deformation and stress analysis of orthogonally stiffened cylindrical shell subjected to static internal pressure. The shell, stringers, and rings are treated as separate structural components. The interactions among the elements are coupled by requiring compatible deformations of the shell, the stringers and the rings. Series solutions satisfying the governing differential equations of the three basic components are used to solve the problem. The convergence of the resulting series for calculating the deformation of the structure is excellent; however, the convergence of the series solution for calculating higher derivatives of the deformation is sometimes not rapid due to the truncation of terms in the series in actual numerical computations. The quantities which involve higher derivatives of displacements, such as stress couples, are therefore calculated by using a finite-difference technique. The displacements used in these finite-difference expressions are computed first according to the series solution and hence, any desirable mesh size may be used. Small deformation and linear theories are used in the analysis, and the closed end effects are evaluated separately and then superimposed upon the solution corresponding to an open-ended cylinder subjected to normal loading.

Analysis

Cylindrical shells which are orthogonally stiffened by uniform and equally-spaced stringers and rings are considered for analysis. In the modern large vehicle designs, such as the giant C-5A airplane, intermediate circumferential bands are provided between every two adjacent rings of the fuselage for more effective structural performance and fail safe considerations. Nonuniform skin thickness in the longitudinal direction is therefore accounted for. Furthermore, the shell is considered to be long and the effects of the supporting conditions at the remote ends are negligible. As a result, the deformation patterns between every two adjacent stringers and every two adjacent rings are considered to be identical throughout the shell structure, and therefore, only one typical portion needs to be analyzed. The basic structural components involved in the analysis as shown in Fig. 1 are the shell, a stringer and a ring. The boundary conditions at the edges of the components are the vanishing of slope and transverse shear.

The general solutions according to linear theories for each component will first be obtained in terms of the interacting forces. The interaction among the components will then be coupled by requiring compatible deformations along the intersections. The analysis and equations governing the behavior of these components without consideration of closed-end effects are first presented, and the effects of closed ends to the deformation and stresses will be discussed separately. More detailed derivations and discussions may be referred to in Ref. 19.

Open Ended Cylindrical Shells

1 Stringer

The following system of equations are taken directly according to the elementary beam theory

$$EI\left(\frac{d^4W_s}{dx^4}, -\frac{d^3W_s}{dx^3}, -\frac{d^2W_s}{dx^2}\right) = (q, V_s, M_s) \tag{1}$$

where E is the modulus of elasticity, I is the moment of inertia of the cross-sectional area, W is the transverse displacement, V is the transverse shear, and M_s is the bending moment; the subscript s corresponds to the stringer, and the sign convention is shown in Fig. 1. The solution

$$W_s = W_{s0} + \sum_{m=2,4}^{\infty} \frac{q_m}{\alpha_m{}^4 EI} \cos\alpha_m x \tag{2}$$

corresponding to the loading

$$q(x) = \frac{q_0}{2} + \sum_{m=2,4}^{\infty} q_m \cos\alpha_m x \tag{3}$$

is seen to satisfy the differential equation given in Eq. (1) as well as the boundary conditions $W_s'(0) = W_s'(l) = V_s(0) = V_s(l) = 0$. q_m represents the Fourier coefficient of the interacting force q, $\alpha_m = m\pi/l$; W_{s0} corresponds to the rigid body displacement of the stringer. Since the transverse shear forces are zero at $x = 0$ and $x = l$, one may conclude that $q_0/2$, the mean value of loading on the stringer, must be zero.

2 Ring

Since the rings used in fuselage design are quite closely spaced and consequently the depth of rings is small when compared to the radius of the fuselage. Eight inch I-section rings were used in the preliminary design of C-5A airplane fuselage and 6-in. rings are typically used in the final design where the radius of fuselage is 143 in. The depth to radius of curvature ratio falls within the thin shell range commonly accepted. For example, the limits of the ratio given by Novozhilov[6] and Kraus[20] are $\frac{1}{20}$ and $\frac{1}{10}$ respectively. One may consider a thin ring analysis by simply using the governing differential equations for cylindrical shells with terms associated with longitudinal coordinate neglected. These equations may be obtained from many textbooks such as in Ref. 21.† More exact analysis for rings or curved beams given in existing textbooks such as in Ref. 22 should be used if the size of the ring becomes large. The equilibrium equations, stress and displacement relations according to thin ring analysis are

$$dT_r/dy - V_r/R_0 = 0 \tag{4}$$

$$dV_r/dy + T_r/R_0 = -2Q \tag{5}$$

† See Eqs. (300–303), pp. 513.

$$-dM_r/dy + V_r = 0 \quad (6)$$

$$T_r = EA_R[(dv/dy) - (W_r/R_0)] \quad (7)$$

$$M_r = -EI_R[(dv/R_0dy) + (d^2W_r/dy^2)] \quad (8)$$

Some symbols and sign convention are shown in Fig. 1. A_R is the cross-sectional area of the ring, and I_R is the moment of inertia of the cross-sectional area.

By differentiation and elimination, Eqs. (4–8) may be combined into

$$I_R \frac{d^3W_r}{dy^3} - A_R \frac{dW_r}{dy} + R_0A_R \frac{d^2v}{dy^2} + I_R \frac{d^2v}{R_0dy^2} = 0 \quad (9)$$

$$-I_R \frac{d^4W_r}{dy^4} - \frac{A_R}{R_0^2} W_r + \frac{A_R}{R_0}\frac{dv}{dy} - I_R \frac{d^3v}{R_0dy^3} = -\frac{2Q}{E} \quad (10)$$

It may be noted that I_R in general is substantially smaller than R_0A_R in fuselage design. The maximum value of $I_R/(R_0^2A_R)$ in numerical examples presented later is about 0.00043. Also the circumferential displacement v in general is substantially smaller than W_r. Furthermore, the fourth term in the left hand side of Eq. (10) is in general much smaller than the third term. Equations (9) and (10) may be combined into the following governing differential equation

$$\beta_1^* \frac{d^5W_r}{dy^5} + \beta_2^* \frac{d^3W_r}{dy^3} + \beta_3^* \frac{dW_r}{dy} = 2\frac{dQ}{dy} \quad (11)$$

where

$$\beta_1^* = EI_R[1 - (I_R/R_0^2A_R\alpha^*)]$$
$$\beta_2^* = (2EI_R/R_0^2\alpha^*)$$
$$\beta_3^* = (EA_R/R_0^2)[1 - (1/\alpha^*)] \quad (12)$$

and where

$$\alpha^* = 1 + (I_R/A_RR_0^2)$$

If $(I_R/A_RR_0^2)$ is neglected when compared to 1, one obtains

$$\beta_1^* = EI_R - (EI_R^2/R_0^2A_R),\ \beta_2^* = (2EI_R/R_0^2) \text{ and } \beta_3^* = 0 \quad (13)$$

Identical results will be obtained if one neglects the last terms on the left hand sides of Eqs. (9) and (10).

The loading function, $2Q(y)$, is considered in the series form

$$2Q = Q_0 + \sum_{n=2}^{\infty} 2Q_n \cos\frac{n\pi y}{b} \quad (14)$$

Since Q_0 represents the average uniform loading intensity on the ring, hence, the average ring deformation becomes

$$W_{r0} = (Q_0aR_0/EA_R) \quad (15)$$

In order to satisfy the governing differential Eq. (11) and the vanishing of slope and transverse shear at $y = 0$ and $y = b$, the following solution is obtained:

$$W_r = \frac{Q_0aR_0}{EA_R} + \sum_{n=2,4}^{\infty} \frac{2Q_n}{(\beta_1^*\beta_n^4 - \beta_2^*\beta_n^2 + \beta_3^*)} \cos\beta_n y \quad (16)$$

Other quantities T_r, M_r and V_r may be obtained from Eqs. (6–8) and in conjunction with Eq. (16).

3 Shell

A typical portion of the shell under investigation is shown in Fig. 1. The origin is chosen at midway between two rings and midway between two stringers. Linear bending theory for a cylindrical shell with nonuniform skin thickness along x direction is considered. For convenience, tensor notation is used and the system of basic equations corresponding to normal loading are presented below. Similar derivation for uniform shell may be seen in Refs. 23, 24

$$N_{ij,j} = 0 \quad (17)$$

$$Q_{i,i} + K_{ij}N_{ij} - P = 0 \quad (18)$$

$$Q_i = M_{ij,j} \quad (19)$$

$$(1/t)N_{ij} = (E\nu/1 - \nu^2)e_{kk}\delta_{ij} + 2\mu e_{ij} \quad (20)$$

$$e_{ij} = \frac{1+\nu}{Et} N_{ij} - \frac{\nu}{Et} N_{kk}\delta_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i} - 2K_{ij}W) \quad (21)$$

$$M_{ij} = -\tfrac{1}{2}D[2\nu X_{kk}\delta_{ij} + (1-\nu)(X_{ij} + X_{ji})] \quad (22)$$

$$X_{ij} = W_{,ij} \text{ and } K_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 1/a \end{bmatrix} \quad (23)$$

Various symbols shown in Eqs. (17–23) are defined under Symbols. The relevant compatibility equation is

$$\epsilon_{ia}\epsilon_{jb}e_{ij,ab} = \epsilon_{ia}\epsilon_{jb}K_{ij}W_{,ab} \quad (24)$$

where ϵ_{ia} is the permutation symbol.

Substitution of Eq. (21) into Eq. (24) and introducing into the resulting equation the Airy stress function, F, such that

$$N_{ij} = \nabla^2F\delta_{ij} - F_{,ij} \quad (25)$$

one obtains

$$\triangledown^4F = -(1/a)W_{,11} \quad (26)$$

where

$$\triangledown^4 = \frac{1}{Et}\left(\frac{\partial^4}{\partial y^4} - \nu\frac{\partial^4}{\partial x^2\partial y^2}\right) + \frac{\partial^2}{\partial x^2}\left[\frac{1}{Et}\left(\frac{\partial^2}{\partial x^2} - \nu\frac{\partial^2}{\partial y^2}\right)\right] + 2(1+\nu)\frac{\partial}{\partial x}\left(\frac{1}{Et}\frac{\partial^3}{\partial x\partial y^2}\right) \quad (27)$$

Elimination of Q_i between Eqs. (18) and (19) and substitution of Eqs. (22) and (25) into the resulting equation leads to the following equation:

$$\bar{\nabla}^4W = (1/a)F_{,11} - P \quad (28)$$

where

$$\bar{\nabla}^4 = D[(\partial^4/\partial y^4 + \nu(\partial^4/\partial x^2\partial y^2))] + (\partial^2/\partial x^2)\{D[\partial^2/\partial x^2 + \nu(\partial^2/\partial y^2)]\} + 2(1-\nu)(\partial/\partial x)[D(\partial^3/\partial x\partial y^2)] \quad (29)$$

and

$$P = p + q(x)\delta(y - b/2) + 2Q(y)\delta(x - l/2) \quad (30)$$

where $\delta(y - y_0)$ is the singularity function.

Equations (26) and (28) are the governing differential equation of the shell. The constant, 2, used in last term of Eq. (14) is merely for convenience.

For a uniform cylindrical shell stiffened by intermediate uniform circumferential bands, the extensional stiffness, $1/Et$, and the flexural rigidity, D, of the shell are

$$(D, 1/Et) = \sum_{i=1}^{\infty} (D, 1/Et_i)[u(x - l_{i-1}) - u(x - l_i)] \quad (31)$$

where N depends on the number of bands, and $u(x - x_0)$ is the unit stepfunction. $\triangledown^4$ and $\bar{\nabla}^4$ reduce to the well-known biharmonic operator ∇^4 for a shell having uniform skin thickness, i.e.,

$$Et\triangledown^4 = (1/D)\bar{\nabla}^4 = \nabla^4 \quad (32)$$

The zero slope and vanishing of transverse shear as boundary conditions involve odd derivatives of normal displacement with respect to x and y along the edges $x = 0,l$ and

$y = 0,b$, respectively. These boundary conditions suggest that the general solution of Eqs. (26) and (28) may be taken in the following double cosine series form:

$$W = \sum_{m=0,2}^{\infty} \sum_{n=0,2}^{\infty} W_{mn} \cos\alpha_m x \cos\beta_n y \qquad (33)$$

and

$$F = \sum_{m=0,2}^{\infty} \sum_{n=0,2}^{\infty} F_{mn} \cos\alpha_m x \cos\beta_n y - \frac{1}{a} \iint W_{00} Et dx dx \qquad (34)$$

where

$$\alpha_m = m\pi/l \text{ and } \beta_n = n\pi/b \qquad (35)$$

m and n take on even integers because of symmetry about $x = l/2$ and $y = b/2$. Therefore, the region bounded by $x = 0,l/2$ and $y = 0,b/2$ only need be considered.

Since the series solution will be used in the analysis, the loading function shown in Eqs. (28) and (30) will be expanded into appropriate series as follows:

$$P = \sum_{m=2,4}^{\infty} \sum_{n=2,4}^{\infty} P_{mn} \cos\alpha_m x \cos\beta_n y + \sum_{n=2,4}^{\infty} P_{0n} \cos\beta_n y + \sum_{m=2,4}^{\infty} P_{m0} \cos\alpha_m x + P_{00} \qquad (36)$$

where

$$\begin{aligned} P_{mn} &= (4Q_n/l) \cos(m\pi/2) + (2q_m/b) \cos(n\pi/2) \\ P_{0n} &= (q_0/b) \cos(n\pi/2) + 2Q_n/l \\ P_{m0} &= q_m/b + (2Q_0/l) \cos(m\pi/2) \\ P_{00} &= q_0/2b + Q_0/l + p \end{aligned} \qquad (37)$$

Note that the following relationship

$$\int_0^l f(\xi)\delta(\xi - \xi_0) d\xi = f(\xi_0) \qquad (38)$$

has been used.

The solution corresponding to a shell structure with uniform skin may be obtained directly from the general solution for shells with intermediate circumferential straps. However, unnecessary errors will be inherited in such limiting case because of series expansions of skin stiffnesses used in the analysis. Therefore, a separate solution for a shell with uniform skin thickness will be presented first and a general solution corresponding to shells with straps will be presented subsequently.

4 Shell With Uniform Skin

For this case, the linear operator shown in Eq. (32) will be used in the governing differential Eqs. (26) and (28). The last term in Eq. (34) will simply be $(Et/2a)W_{00}$. By substituting Eqs. (33, 34, and 36) in conjunction with Eqs. (32, 36, and 37) into Eqs. (26) and (28), and by collecting like terms in the results, one may express the coefficients of w in terms of loading coefficients as follows:

$$W_{mn} = -\frac{\alpha_{mn}^4}{D\alpha_{mn}^8 + (Et/a^2)\alpha_m^4} \left(\frac{4Q_n}{l} \cos\frac{m\pi}{2} + \frac{2q_m}{b} \cos\frac{n\pi}{b} \right),$$

$$W_{0n} = -\frac{1}{D\beta_n^4 + Et/a^2} \left(\frac{q_m}{b} + \frac{2Q_0}{l} \cos\frac{m\pi}{2} \right),$$

$$W_{00} = \frac{a^2}{Et} \left(p + \frac{Q_0}{l} \right) \qquad (39)$$

where

$$\alpha_{mn}^2 = \alpha_m^2 + \beta_n^2 \qquad (40)$$

Expressions similar to Eqs. (39) may also be obtained for the coefficient of F.‡

By matching the displacements between the shell and stringer along the intersecting line according to Eqs. (2) and (33) in conjunction with Eq. (39), one obtains the following relationships:

$$\sum_n^{\infty} \frac{2\cos(n\pi/2)}{D\beta_n^4 l} Q_n = W_{00} = \frac{a^2}{Et} \left(p + \frac{Q_0}{l} \right) \qquad (41)$$

and

$$\sum_n^{\infty} \frac{4\alpha_{mn}^4 \cos(n\pi/2) \cos(m\pi/2)}{[D\alpha_{mn}^8 + (Et/a^2)\alpha_m^4] l} Q_n + \frac{2\cos(m\pi/2)}{[D\alpha_m^4 + (Et/a^2)]l} Q_0 + \frac{q_m}{b} \left[\frac{b}{EI\alpha_m^4} + \frac{1}{D\alpha_m^4 + Et/a^2} + \sum_n^{\infty} \frac{2\alpha_{mn}^4 \cos^2(n\pi/2)}{D\alpha_{mn}^8 + (Et/a^2)\alpha_m^4} \right] = 0 \qquad (42)$$

In a similar manner, the following relationships are obtained according to Eqs. (16) and (33) in conjunction with Eq. (39) in order to satisfy the condition for compatible deformation between the skin and the ring,

$$-\sum_{m=2}^{\infty} \gamma_{mn} \cos\frac{m\pi}{2} \left(\frac{4Q_n}{l} \cos\frac{m\pi}{2} + \frac{2}{b} q_m \cos\frac{n\pi}{2} \right) - \frac{2}{D\beta_n^4 l} Q_n = [\beta_1^* \beta_n^4 - \beta_2^* \beta_n^2 + \beta_3^*]^{-1} (2Q_n) \qquad (43)$$

and

$$W_{00} = \frac{a^2}{Et} \left(p + \frac{Q_0}{l} \right) = \left[\frac{a^2}{EA_R} + \sum_{m=2}^{\infty} \frac{2}{(D\alpha_m^4 + Et/a^2)l} \right] Q_0 + \sum_{m=2}^{\infty} \eta_m q_m \qquad (44)$$

where

$$\gamma_{mn} = [\alpha_{mn}^4 / D\alpha_{mn}^8 + (Et/a^2)\alpha_m^4] \qquad (45)$$

$$\eta_m = [\cos(m\pi/2)/(D\alpha_m^4 + Et/a^2)b] \qquad (46)$$

Q_0, Q_n and W_{00} are eliminated in Eqs. (41–44) and a system of an infinite number of simultaneous equations results,

$$[A_{mj}][q_j] = [C_m] \qquad (47)$$

where

$$A_{mj} = \sum_{n=2}^{\infty} \left(\frac{4}{l} \cos\frac{m\pi}{2} \cos\frac{j\pi}{2} \gamma_{mn}\gamma_{jn}\zeta_n \right) + \epsilon_j \delta_{jm} + g_m \eta_j,$$

$$C_m = -g_m \frac{a^2}{Et}, \quad g_m = -\frac{2\cos(m\pi/2)}{(D\alpha_m^4 + Et/a^2)l} \times \left[\left(\frac{1}{Etl} + \frac{1}{EA_R} \right) a^2 + \sum_j^{\infty} \frac{2}{(D\alpha_j^4 + Et/a^2)l} \right]^{-1},$$

$$\zeta_n = -\frac{1}{b} \left[\frac{1}{D\beta_n^4 l} + \frac{1}{\beta_m^2(\beta_1^*\beta_n^2 - \beta_2^*)} + \sum_{j=2}^{\infty} \gamma_{jn} \left(\frac{2}{l} \right) \right]^{-1},$$

$$\epsilon_j = \frac{1}{(D\alpha_j^4 + Et/a^2)b} + \frac{1}{EI\alpha_j^4} + \sum_k^{\infty} \gamma_{jk} \left(\frac{2}{b} \right) \qquad (48)$$

δ_{mj} is the Kronecker delta.

In practice, only a finite number of Eqs. (47) will be considered. After having solved for $q_1, q_2, \ldots q_m$, the quantities Q_n, P_{mn}, W_{mn}, W_{on}, Q_o, W_{mo}, W_{oo}, $W(x,y)$ and N_{yy} can be subsequently calculated, and the detailed expressions may be referred to in Ref. 19.

‡ Detailed expressions may be referred to in Ref. 19.

Since only a finite number of terms is taken in the series which accurately represents the displacement W, there is no assurance of the convergence of the higher derivatives of this truncated series. Therefore, the stress couple M_{ij}, bending moments M_s and M_r, transverse shears Q_x, Q_y, V_s and V_r, and the ring stress resultants T_r are calculated by using a finite-difference technique. Central difference representation of derivatives is used. The displacements for finite difference expressions are calculated according to the series solution presented previously in the analysis whenever and wherever needed.

5 Shell with Straps

For this case, the skin is considered to have nonuniform thickness. The variation of the skin thickness takes the form shown in Eq. (31). Substitution of Eqs. (33, 34, and 36) into Eqs. (26) and (28) in conjunction with the series expressions§

$$\left(Et, \frac{1}{Et}, \frac{1}{Et} \cos\alpha_m x\right) = \frac{1}{2}(\lambda_{0n0}, \delta_{0n0}, \delta_{mn0}) + \sum_{r=2}^{\infty} (\lambda_{0ns}, \delta_{0ns}, \delta_{mns}) \cos\alpha_s x \quad (49)$$

$$\left(\frac{1}{Et}, D\right) \sin\alpha_m x = \sum_{r}^{\infty} (\Delta_{mns}, E_{mns}) \sin\alpha_s x \quad (50)$$

$$(D, D\cos\alpha_m x) = \frac{1}{2}(\epsilon_{0n0}, \epsilon_{mn0}) + \sum_{r=2}^{\infty} (\epsilon_{0ns}, \epsilon_{mns}) \cos\alpha_s x \quad (51)$$

results in the following equations

$$\sum_{m}^{\infty} F_{mn}(\beta_n^4 - \nu\alpha_m^2\beta_n^2) \frac{\delta_{mn0}}{2} + \beta_n^4 \frac{\delta_{0n0}}{2} F_{0n} = 0 \quad (52)$$

$$\sum_{m}^{\infty} F_{mn}\{[(\beta_n^4 - \nu\alpha_m^2\beta_n^2) + (\alpha_m^2 - \nu\beta_n^2)\alpha_s^2]\delta_{mns} + 2(1+\nu)\alpha_m\alpha_s\beta_n^2\Delta_{mns}\} + F_{0n}(\beta_n^4 - \nu\beta_n^2\alpha_s^2)\delta_{0ns} = \frac{1}{a} W_{sn}\alpha_s^2 \quad (53)$$

$$\sum_{m}^{\infty} F_{n0}\alpha_m^2\alpha_s^2\delta_{m0s} = (1/a) W_{s0}\alpha_s^2 \quad (54)$$

$$\sum_{m}^{\infty} W_{mn}(\beta_n^4 + \nu\alpha_n^2\beta_n^2) \frac{\epsilon_{mn0}}{2} + \beta_n^4 \frac{\epsilon_{0n0}}{2} W_{0n} = -\frac{2Q_n}{l} \quad (55)$$

$$\sum_{m}^{\infty} W_{mn}\{[(\beta_n^4 + \nu\alpha_m^2\beta_n^2) + (\alpha_m^2 + \nu\beta_n^2)\alpha_s^2]\epsilon_{mns} + 2(1-\nu)\alpha_m\beta_n^2\alpha_s E_{mns}\} + W_{0n}(\beta_n^4 + \nu\beta_n^2\alpha_s^2)\epsilon_{0ns} = -\frac{1}{a}\alpha_s^2 F_{sn} - \frac{4Q_n}{l}\cos\frac{s\pi}{2} - \frac{2q_r}{b}\cos\frac{n\pi}{2} \quad (56)$$

$$\sum_{m}^{\infty} W_{m0}\alpha_m^2\alpha_s^2\epsilon_{m0s} = -\frac{1}{a}\alpha_s^2 F_{s0} - \frac{2Q_0}{l}\cos\frac{s\pi}{2} - \frac{q_r}{b} - \frac{W_{00}}{a^2}\lambda_{0ns} \quad (57)$$

$$Q_0/l + p + (W_{00}/2a^2)\lambda_{0n0} = 0 \quad (58)$$

By requiring compatible deformations of the skin, the stringer, and the ring, i.e.,

$$w[x, (b/2)] = W_s \quad (59)$$

and

$$w[(l/2), y] = W_r \quad (60)$$

§ Detailed expressions for Fourier coefficients may be referred to in Ref. 19.

Equations (2, 16, and 33) in conjunction with Eqs. (52–58) are substituted into Eqs. (59) and (60). After a lengthy mathematical manipulation,¶ the following system of relationships is obtained,

$$\sum_{j}^{M}\sum_{m}^{M}\left(D_{mns}\frac{a}{\alpha_m^2}\left\{e_{jnm} - \beta_n^2(\beta_n^2 + \nu\alpha_m^2)\epsilon_{0nm}\cos\frac{j\pi}{2} - A_n\beta_n^2\left[(\beta_n^2 + \nu\alpha_j^2)\epsilon_{jn0} - \beta_n^2\epsilon_{0n0}\cos\frac{j\pi}{2}\right]\times\left[\frac{\beta_n^2 + \nu\alpha_m^2}{\beta_1{}^*\beta_n^2 - \beta_2{}^*}\epsilon_{0nm} + \frac{2}{l}\cos\frac{m\pi}{2}\right]\right\} + \frac{1}{a}\alpha_s^2\delta_{sj}\right)W_{jn} + \sum_{k}^{K}\sum_{m}^{M} D_{mns}\left(\frac{2a}{b}\right)\cos\frac{n\pi}{2} EI\alpha_m^2\cos\frac{k\pi}{2} W_{mk} + \sum_{m}^{M} D_{mns}\left(\frac{2a}{b}\right)\cos\frac{n\pi}{2} EI\alpha_m^2 W_{m0} = 0 \quad (61)$$

$$\sum_{j}^{M}\sum_{m}^{M}\left[\alpha_s^2\alpha_m^2\delta_{m0s}\left\{\alpha_j^2\epsilon_{j0m} + \frac{EI}{b}\alpha_m^2\delta_{mj} + \frac{1}{a^2}\beta_m\cos\frac{j\pi}{2}\right\} + \frac{\alpha_s^2}{a^2}\delta_{sj}\right]W_{j0} + \sum_{m}^{M}\sum_{k}^{K}\alpha_s^2\alpha_m^2\delta_{m0s}\times\frac{EI}{b}\cos\frac{k\pi}{2}W_{mk} = p\alpha_s^2\sum_{m}^{M}\alpha_m^2\delta_{m0s}\left[\frac{2}{\lambda_{0n0}}\beta_n + \frac{\lambda_{0nm}}{\alpha_m^2\lambda_{0n0}}\right] \quad (62)$$

where

$$A_n = \left(\frac{2}{l} + \epsilon_{0n0}\beta^2\frac{1}{\beta_1{}^*\beta_n^4 - \beta_2{}^*\beta_n^2 + \beta_3{}^*}\right)^{-1}$$

$$D_{mns} = \delta_{mns}[\beta_n^4 - \nu\alpha_m^2\beta_n^2 + (\alpha_m^2 - \nu\beta_n^2)\alpha_s^2] + 2(1+\nu)\alpha_m\alpha_s\beta_n^2\Delta_{mns} - (\beta_n^2 - \nu\alpha_m^2)(\beta_n^2 - \nu\alpha_s^2)\frac{\delta_{mn0}\delta_{0ns}}{\delta_{0n0}},$$

$$e_{jnm} = \epsilon_{jnm}[\beta_n^4 + \nu\alpha_j^2\beta_n^2 + (\alpha_j^2 + \nu\beta_n^2)\alpha_m^2] + 2(1-\nu)\alpha_j\alpha_m\beta_n^2E_{jnm},$$

$$B_m = \frac{2}{\alpha_m^2 l}\left(\cos\frac{m\pi}{2} - \frac{\lambda_{0nm}}{\lambda_{0n0}}\right)\left(\frac{2}{\lambda_{0n0}l} + \frac{1}{EA_R}\right)^{-1} \quad (63)$$

Equations (61) and (62) represent a set of infinitely many simultaneous algebraic equations. In the actual numerical calculation, only a finite number of terms will be considered. After W_{mn} are determined, the displacements and internal loads will be calculated in the same manner as discussed in the previous case.

6 Closed End Effects

When the cylindrical shell subjected to internal pressure is closed at both ends, a total axial load of $\bar{P} = \pi a^2 p$ will be carried by the stringers and skins. The stresses due to $\bar{P}$ are small when compared to the effects due to lateral pressure alone. The following analysis without consideration of shear lag effect will be quite acceptable.

It is well known that the normal displacement of a thin cylinder, without consideration of constraints of rings and stringers, subjected to lateral pressure alone is

$$W_0 = pa^2/Et \quad (68)$$

The corresponding final deformations, considering the effect of the constraints of the ring and the stringer, may be written as

$$w = Wpa^2/Et \quad (69)$$

The reduction or addition of deformation, as well as stresses

¶ Detailed derivation may be referred to in Ref. 19.

Table 1 Data for numerical examples

Case	a in.	b in.	R_0 in.	I in.^{4}	A_s in.^{2}	A_R in.^{2}	I_R in.^{4}	t in.	$t_s(Ti)$ in.	$l_s = 2l_1$ in.	Remark
1	143	7.8	138.9	0.051	0.1881	0.721	6.56	0.07	0	0	8 in. ring without strap
2	143	7.8	138.9	0.051	0.1881	0.721	6.56	0.07	0.02	4.5	8 in. ring with strap
3	143	7.8	138.9	0.051	0.1881	0.600	3.92	0.07	0.02	4.5	6 in. ring with strap

due to the constraints of the stringers and rings, may be indicated by the parameter, $\alpha(x,y)$, or

$$-\alpha(x,y) = (w - W_0)/|W_0| = W - 1 \qquad (70)$$

Substitution of Eq. (68) into (69) yields

$$w = (1 - \alpha)W_0 = (1 - \alpha)pa^2/Et \qquad (71)$$

The state of stress and deformation, $\bar{w}$, without consideration of the constraints of the ring and the flexural constraint of the stringer due to axial load $\bar{P}$ or $pa/2$ intensity along the circumference, may be derived by satisfying the equilibrium conditions and the compatibility of linear strain of the skin and the stringer. The following results are obtained:

$$\bar{\sigma}_{xx} = pa/2(1 + A_s/bt)t \qquad (72)$$

and

$$N_{xs} = \bar{\sigma}_{xx}A_s = paA_s/2t(1 + A_s/bt) \qquad (73)$$

The corresponding circumferential stress is

$$\bar{\sigma}_{yy} = 0$$

As a result, the contraction of the skin in the radial direction corresponding to zero hoop stress becomes

$$\bar{w} = \nu pa^2/2Et(1 + A_s/bt) \qquad (74)$$

The increase or decrease in the displacement due to constraints of the ring and the flexural rigidity of the stringer is $\alpha\bar{w}$. The parameter, α, has been defined in Eq. (70). The final displacement, w^*, of the skin due to the axial load $\bar{P}$ becomes

$$w^* = (1 - \alpha)\bar{w} = [\nu/2(1 + A_s/bt)](1 - \alpha)(pa^2/Et) \qquad (75)$$

The corresponding circumferential stress, $\sigma_{yy}{}^*$, becomes

$$\sigma_{yy}{}^* = \alpha E(\bar{w}/a) = [\alpha\nu/2t(1 + A_s/bt)]pa = \alpha\nu_x\sigma_x{}^* \qquad (76)$$

which is considerably smaller than the hoop stress corresponding to lateral pressure alone.

From the above discussion, it is interesting to note that the deformation and stresses due to the effect of axial load may be determined as soon as the solution for an open-ended cylindrical shell is obtained.

Numerical Examples

Typical arrangements and dimensions of C-5A airplane fuselage are selected for example solutions. The material parameters and dimensions common to all cases listed in Table 1 are $p = 1$ psi, $l = 20$ in., $\nu = 0.3$, $E = 10^7$ psi, $\Delta x = \Delta y = 0.25$ in. The symbol, t_s, shown in the table represents the thickness of a titanium strap and l_s is its width. The straps are placed symmetrically along the mid-line between two adjacent rings.

The results for the displacements and inner and outer fiber stresses corresponding to the combined effect of lateral pressure and axial load at the following locations are plotted in Figs. 2–4: 1) along mid-line between adjacent stringer

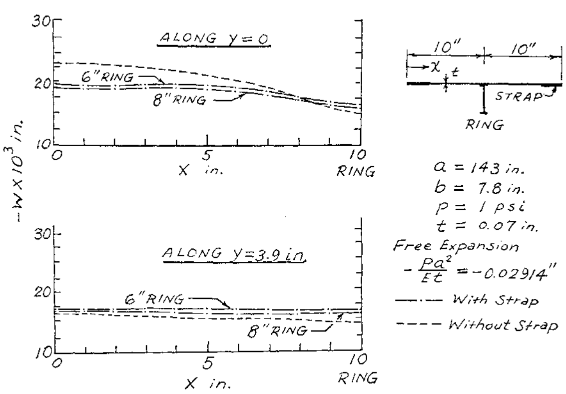

Fig. 2a Skin displacements along longitudinal direction.

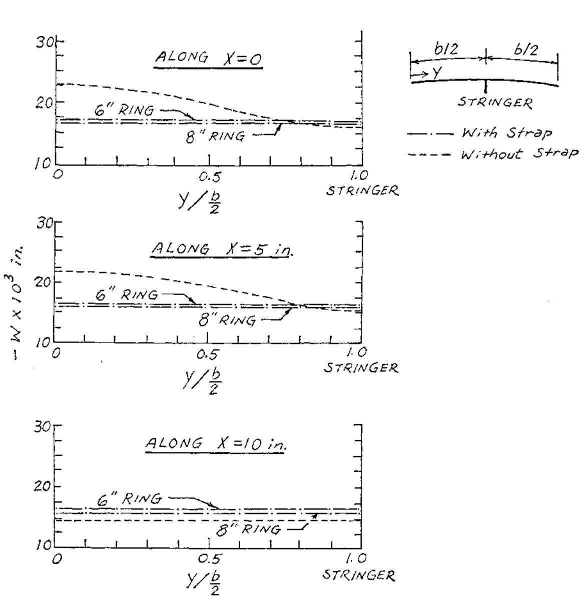

Fig. 2b Skin displacements along circumferential direction.

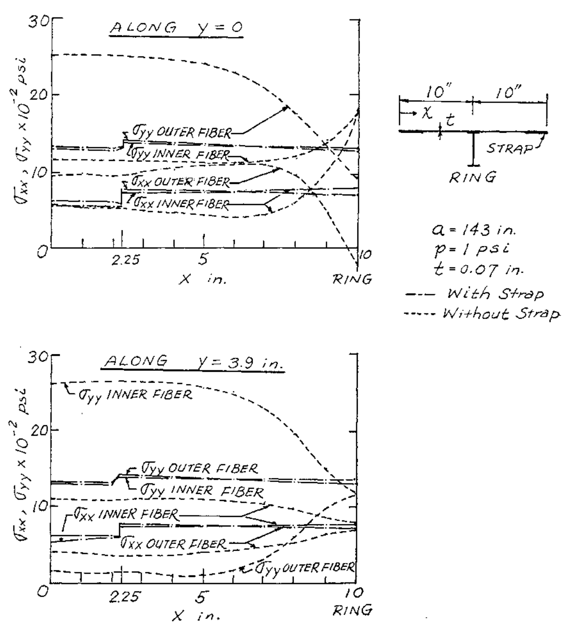

Fig. 3 Skin longitudinal and circumferential stresses at points along longitudinal direction.

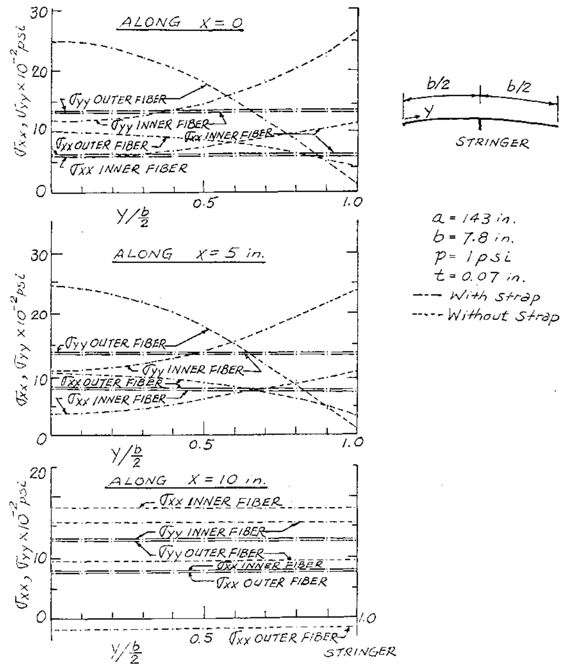

Fig. Skin longitudinal and circumferential stress at points along circumferential direction.

($y = 0$), 2) along stringer ($y = b/2$), 3) along mid-line between two adjacent rings ($x = 0$), 4) along quarter-line of ring spacing ($x = l/4$), 5) along center-line of ring ($x = l/2$).

The variation of displacement of the skin along the longitudinal as well as the circumferential directions is significant for the cases where circumferential bands are not included; as a result, the bending moments are high. For a shell with intermediate straps, however, the variation of displacements and stresses along the circumferential direction is negligibly small, and the variations along longitudinal direction are small but detectable. In short, it appears that the insertion of straps is structurally effective according to the cases under consideration.

Discussion

The solution of cylindrical shells with uniform as well as nonuniform thickness orthogonally stiffened by uniform stiffeners subjected to internal pressure is obtained by treating the stiffeners as separate elements. The deformation and stress condition can be calculated for any point in the structure. Numerical examples based on C-5A fuselage material parameters and dimensions are presented. The results indicate that the variation of displacement of the shell along the longitudinal as well as the circumferential directions is significant for the cases where the circumferential bands are not included; as a result, large bending stresses are introduced. For a shell with intermediate bands, however, the variation of displacements and stresses along the circumferential direction is negligibly small, and the variations along longitudinal directions are small but detectable. As a result, stress levels are equalized and maximum stress levels are reduced. This reduction will increase the fatigue endurance where fuselage pressurization and de-pressurization are predominant, and in resistance to damage propagation. In short, it appears that the insertion of circumferential bands is structurally effective according to the cases considered. While the example problems consider only one strap located midway between every two rings, the analysis and computer program for IBM 7094 are prepared for multiple straps at variable locations.

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